ON THE STABILITY OF PERIODIC SOLUTIONS OF NONAUTONOMOUS QUASILINEAR SYSTEMS WITH ONE DEGREE OF FREEDOM

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The stability of periodic solutions for the case of double roots of the equation of fundamental amplitudes is investigated in this article.

The nonautonomous quasilinear system

$$\ddot{x} + m^2 x = f(t) + \mu F(t, x, \dot{x}, \mu) \qquad (1)$$

is considered.

Here μ is a small positive parameter; f is a continuous periodic function of time t with period 2π. It is assumed that the Fourier expansion of f does not contain harmonics of order m (m is an integer). F is an analytic function of the variables x, $\dot{x}$, μ, and it is periodic, of period 2π, in t. The resulting solution

$$x_0(t) = A_0 \cos mt + \frac{B_0}{m} \sin mt + \varphi(t)$$

depends on two arbitrary constants A_0 and B_0; the function $\varphi(t)$ describes the forced oscillation of the generating system ($\mu = 0$).

The periodic solution of period 2π of equation (1), which transforms into the solution $x_0(t)$ when $\mu = 0$, is found by the method of Poincaré [1-3]. The initial conditions used are

$$x(0) = x_0(0) + \beta_1, \qquad \dot{x}(0) = \dot{x}_0(0) + \beta_2$$

Here β_1 and β_2 are functions of μ, which vanish when $\mu = 0$.

The amplitudes A_0 and B_0 are determined by means of the equations for the fundamental amplitudes

$$C_1(2\pi) = -\frac{1}{m}\int_0^{2\pi} F(t, x_0, \dot{x}_0, 0)\sin mt dt = 0 \quad \dot{C}_1(2\pi) = \int_0^{2\pi} F(t, x_0, \dot{x}_0, 0)\cos mt dt = 0 \quad (2)$$

If the equations (2) have double roots, the following relations hold

$$\Delta = \begin{vmatrix} \frac{\partial C_1}{\partial A_0} & \frac{\partial C_1}{\partial B_0} \\ \frac{\partial \dot{C}_1}{\partial A_0} & \frac{\partial \dot{C}_1}{\partial B_0} \end{vmatrix} = 0, \qquad \Delta^* = \frac{\partial C_1}{\partial B_0}\begin{vmatrix} \frac{\partial \Delta}{\partial A_0} & \frac{\partial \Delta}{\partial B_0} \\ \frac{\partial \dot{C}_1}{\partial A_0} & \frac{\partial \dot{C}_1}{\partial B_0} \end{vmatrix} \neq 0 \quad (3)$$

Here, all the derivatives are taken when $t = 2\pi$. Under the conditions (3), to the double root A_0, B_0 of (2) there correspond two solutions of equation (1), which were constructed in [3]

$$x^{(k)}(t) = \sum_{n=1}^{\infty} x_{n/2}^{(k)}(t)\,\mu^{n/2} \qquad (k = 1, 2) \quad (4)$$

where the $x_{n/2}^{(k)}(t)$ are periodic functions of time. The expression (4) includes also the case when the solutions are expressed as series in integral powers of μ.

Let us investigate the stability of the periodic solutions of (4). The equation of variations for equation (1) is

$$\ddot{y} + m^2 y - \mu\left(\frac{\partial F}{\partial \dot{x}}\right)_k \dot{y} - \mu\left(\frac{\partial F}{\partial x}\right)_k y = 0 \quad (5)$$

The subscript k at the parentheses indicates that one must replace $\dot{x}$ and x by $x^{(k)}(t)$ and $\dot{x}^{(k)}(t)$ in the derivatives of the function F.

As is known (see, e.g. [1, p.163]), the functions

$$y_1^{(k)}(t) = \frac{\partial x^{(k)}(t)}{\partial A_0}, \qquad y_2^{(k)}(t) = \frac{\partial x^{(k)}(t)}{\partial B_0}$$

form a fundamental system of the equation (5) with the boundary conditions

$$y_1^{(k)}(0) = 1, \qquad \dot{y}_1^{(k)}(0) = 0, \qquad y_2^{(k)}(0) = 0, \qquad \dot{y}_2^{(k)}(0) = 1$$

In order that the periodic solutions of equation (1) may be asymptotically stable it is necessary and sufficient that the next two inequalities [1, pp.67-73] be satisfied

$$y_1^{(k)}(2\pi)\,\dot{y}_2^{(k)}(2\pi) - y_2^{(k)}(2\pi)\,\dot{y}_1^{(k)}(2\pi) - y_1^{(k)}(2\pi) - \dot{y}_2^{(k)}(2\pi) + 1 > 0 \quad (6)$$

$$\int_0^{2\pi} \frac{\partial F(t,\ x_0,\ \dot{x}_0,\ 0)}{\partial \dot{x}_0}\, dt < 0 \tag{7}$$

In accordance with (4)

$$y_1^{(k)}(t) = \sum_{n=1}^{\infty} \frac{\partial x_{n/2}^{(k)}(t)}{\partial A_0}\mu^{n/2}, \qquad y_2^{(k)}(t) = \sum_{n=1}^{\infty} \frac{\partial x_{n/2}^{(k)}(t)}{\partial B_0}\mu^{n/2} \tag{8}$$

The condition (6) can be written in the form

$$L_2^{(k)}\mu^2 + L_{5/2}^{(k)}\mu^{5/2} + L_3^{(k)}\mu^3 + L_{7/2}^{(k)}\mu^{7/2} + \cdots > 0 \tag{9}$$

where the $L_{n/2}^{(k)}$ are expressed in terms of $\partial x_{n/2}^{(k)}/\partial A_0$, $\partial x_{n/2}^{(k)}/\partial B_0$, and their derivatives with respect to time when $t = 2\pi$. Substituting the values of these expressions as given in [3], we obtain

$$L_2^{(k)} = \Delta, \qquad L_{5/2}^{(k)} = A_{1/2}\frac{\partial \Delta}{\partial A_0} + B_{1/2}\frac{\partial \Delta}{\partial B_0}$$

$$L_3^{(k)} = A_1\frac{\partial \Delta}{\partial A_0} + B_1\frac{\partial \Delta}{\partial B_0} + \frac{1}{2}A_{1/2}^2\frac{\partial^2 \Delta}{\partial A_0^2} + \frac{1}{2}B_{1/2}^2\frac{\partial^2 \Delta}{\partial B_0^2} + A_{1/2}B_{1/2}\frac{\partial^2 \Delta}{\partial A_0 \partial B_0} - \left(\frac{\partial \Delta_1}{\partial A_0} + \frac{\partial \Delta_2}{\partial B_0}\right)$$

$$\begin{aligned} L_{7/2}^{(k)} = {} & A_{3/2}\frac{\partial \Delta}{\partial A_0} + B_{3/2}\frac{\partial \Delta}{\partial B_0} + A_{1/2}A_1\frac{\partial^2 \Delta}{\partial A_0^2} + B_{1/2}B_1\frac{\partial^2 \Delta}{\partial B_0^2} + (A_{1/2}B_1 + B_{1/2}A_1)\frac{\partial^2 \Delta}{\partial A_0 \partial B_0} + \\ & + \frac{1}{6}A_{1/2}^3\frac{\partial^3 \Delta}{\partial A_0^3} + \frac{1}{2}A_{1/2}^2 B_{1/2}\frac{\partial^3 \Delta}{\partial A_0^2 \partial B_0} + \frac{1}{2}A_{1/2}B_{1/2}^2\frac{\partial^3 \Delta}{\partial A_0 \partial B_0^2} + \frac{1}{6}B_{1/2}^3\frac{\partial^3 \Delta}{\partial B_0^3} - \\ & - A_{1/2}\left(\frac{\partial^2 \Delta_1}{\partial A_0^2} + \frac{\partial^2 \Delta_2}{\partial A_0 \partial B_0}\right) - B_{1/2}\left(\frac{\partial^2 \Delta_1}{\partial A_0 \partial B_0} + \frac{\partial^2 \Delta_2}{\partial B_0^2}\right) \end{aligned}$$

Here we have omitted the index k in $A_{n/2}^{(k)}$, and $B_{n/2}^{(k)}$ for the sake of simplicity; besides we have used the notation

$$\Delta_1 = \frac{\partial C_1}{\partial B_0}\dot{C}_2(2\pi) - \frac{\partial \dot{C}_1}{\partial B_0}C_2(2\pi), \qquad \Delta_2 = \frac{\partial \dot{C}_1}{\partial A_0}C_2(2\pi) - \frac{\partial C_1}{\partial A_0}\dot{C}_2.\ (2\pi)$$

The formulas for $C_2(2\pi)$, $\dot{C}_2(2\pi)$ are given in [2, p.853]. The quantities $A_{n/2}$ and $B_{n/2}$ are the coefficients in the expansion

$$\beta_1 = \sum_{n=1}^{\infty} A_{n/2}^{(k)}\mu^{n/2}, \qquad \beta_2 = \sum_{n=1}^{\infty} B_{n/2}^{(k)}\mu^{n/2}.$$

The inequality (9) is satisfied for sufficiently small μ if the coefficient of the highest order derivative is positive. In case of double roots of $L_2^{(k)} = 0$, one has to examine the sign of the first nonvanishing coefficient. Let us consider the most interesting case [3].

1. Suppose that $\Delta_1 \neq 0$. In view of the fact that $\Delta = 0$, we have

$\Delta_2 \neq 0$. The periodic solutions of (1) are constructed in the form of series in powers of $\mu^{1/2}$ (see [3, p.751]). For them

$$A_{1/2}^{(k)} = \pm \sqrt{E}, \qquad B_{1/2}^{(k)} = \pm \frac{\Delta_2}{\Delta_1} \sqrt{E}, \qquad E = \frac{2\Delta_1}{\Delta^*} \frac{\partial \dot{C}_1}{\partial B_0} \frac{\partial \dot{C}_1}{\partial B_0}. \tag{10}$$

Since we consider only real expansions of the solutions, $E > 0$. In the formulas (10) and in what follows, the upper sign goes with the solution $k = 1$, while the lower sign (the minus sign) goes with the solution when $k = 2$.

As has been shown earlier [3, p.750], it follows from the expression for Δ^* that not a single one of the quantities $\partial C_1/\partial A_0$, $\partial \dot{C}_1/\partial A_0$ $\partial C_1/\partial B_0$, and $\partial \dot{C}_1/\partial B_0$ can vanish. Hence

$$L_{3/2}^{(k)} = \pm \frac{2\Delta_1}{\sqrt{E}} \neq 0$$

and the sign in front of Δ_1 solves the problem on the stability under the fulfillment of the condition (7). If $\Delta_1 > 0$, then the solution for $k = 1$ is stable, while the solution for $k = 2$ is unstable. Conversely, if $\Delta_1 < 0$ the solution for $k = 1$ is unstable, while the solution for $k = 2$ is stable.

2. Suppose that $\Delta_1 = \Delta_2 = 0$. Let us assume also that the real roots a_1 and a_2 of the equation

$$s = N_{02}a^2 + N_{11}a + N_{20} = 0 \tag{11}$$

are simple, i.e. $a_1 \neq a_2$, and, hence

$$2N_{02}a_k + N_{11} \neq 0. \tag{12}$$

In the formulas (11) and (12) the quantities N_{02}, N_{11} and N_{20} are computed with the aid of the formulas (2.7) of the work [3]. For example

$$N_{02} = -\frac{\Delta^*}{2}\left(\frac{\partial C_1}{\partial B_0}\right)^{-2}\left(\frac{\partial \dot{C}_1}{\partial B_0}\right)^{-1} \neq 0.$$

In this case $L_{5/2}^{(k)} = 0$, and the periodic solutions (4) are constructed in the form of series in integral powers of μ. Hereby

$$L_3^{(k)} = -\frac{\partial C_1}{\partial B_0}(2N_{02}a_k + N_{11}) \neq 0$$

in view of (12). For stability it is necessary and sufficient that

$$\frac{\partial C_1}{\partial B_0}(2N_{02}a_k + N_{11}) < 0 \tag{13}$$

Only one of the periodic solutions (4) is stable. Indeed, the left part of condition (12) characterizes the angular coefficient (slope) of the tangent to the parabola (11) at the points a_1 and a_2 where the parabola intersects the axis of the abscissas. If the slope is negative at the point a_1, then it will be positive at a_2, and conversely. Hence, the condition (13) can be fulfilled at only one of the two points a_k.

3. Suppose that the roots of the equation (11) are multiple roots and that the quantity $K \neq 0$ (see [3, p.752]). Obviously, $L_3^{(k)} = 0$. Both periodic solutions can be expressed as series in powers of $\mu^{1/2}$. Hereby

$$L_{7/2}^{(k)} = \mp 2N_{02}\frac{\partial C_1}{\partial B_0}\sqrt{-\frac{K}{N_{02}}}$$

If $N_{02}\partial C_1/\partial B_0 < 0$ then the solution for $k = 1$ is stable, while the solution for $k = 2$ is unstable. Conversely, if $N_{02}\partial C_1/\partial B_0 > 0$ the solution for $k = 2$ is stable, while that for $k = 1$ is unstable.

In the considered cases the condition (6), or the condition (9), is always satisfied for one of the two periodic solutions (4). For other cases, the analysis of the condition (6) is analogous to the preceding one. Besides the condition (6), it is necessary and sufficient for the stability of the solutions (4) that the condition (7) be fulfilled, which can be put into the form

$$\frac{\partial C_1}{\partial A_0} + \frac{\partial \dot{C}_1}{\partial B_0} < 0 \tag{14}$$

Thus, if condition (14) holds, then of the two periodic solutions (4) of equation (1), which correspond to double roots of the equations of the fundamental amplitudes, one and only one will be stable.

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